Student Workbook

Solving Matrix Equations with Inverses

Name: ______________________ Date: ______________ Class: ________

ESSENTIAL QUESTION

How does multiplying both sides of the matrix equation Ax = b by A^{-1} solve the system, and when does this method fail?

Lesson Overview

Write a linear system as Ax = b, where A is the coefficient matrix, x is the variable vector, and b is the constant vector. If A is invertible (det(A) ≠ 0), multiply both sides on the left by A^{-1}: $A^{-1}(Ax) = A^{-1}b \rightarrow Ix = A^{-1}b \rightarrow x = A^{-1}b$.

det(A) → A^{-1} → x = A^{-1}b (Verify A·A^{-1} = I)

2×2 Formula	For A=[[a,b],[c,d]]: det(A)=ad−bc. A^{-1}=(1/det(A))·[[d,−b],[−c,a]]. A^{-1} exists iff det(A)≠0.
Gauss-Jordan	For n×n: form [A|I], row reduce until the left side becomes I — the right side becomes A^{-1}.
Verification	Always check: A·A^{-1} = A^{-1}·A = I.
Method Fails	If det(A)=0, A is singular (not invertible); Ax=b has no unique solution.
Key Advantage	Once A^{-1} is found, solving Ax=b for different b vectors takes only one matrix-vector multiplication each time.

Worked Examples

EXAMPLE 1

Find A^{-1} for A = [[3, 1], [5, 2]].

- det = 3·2 − 1·5 = 1
- A^{-1} = (1/1)·[[2,−1],[−5,3]] = [[2,−1],[−5,3]]
- Verify: A·A^{-1} = [[1,0],[0,1]] ✓

Answer: A^{-1} = [[2, −1], [−5, 3]]

EXAMPLE 2

Solve Ax = b using A^{-1}: A = [[3, 1], [5, 2]], b = [[7], [11]].

- x = A^{-1}b = [[2,−1],[−5,3]] · [[7],[11]]
- x_1 = 2·7+(−1)·11 = 3. x_2 = (−5)·7+3·11 = −2

Answer: x = 3, y = −2

EXAMPLE 3

Find A^{-1} for A = [[1, 2], [3, 4]] using Gauss-Jordan.

- [A|I] = [[1,2|1,0],[3,4|0,1]]. $R_2 \to R_2 - 3R_1$: [[1,2|1,0],[0,−2|−3,1]]
- $R_2 \to R_2 \div (-2)$: [[1,2|1,0],[0,1|3/2,−1/2]]. $R_1 \to R_1 - 2R_2$: [[1,0|−2,1],[0,1|3/2,−1/2]]

Answer: A^{-1} = [[−2, 1], [3/2, −1/2]]

EXAMPLE 4

Determine if A = [[2, 4], [1, 2]] is invertible.

- det(A) = 2·2 − 4·1 = 0
- det = 0 → A is singular, not invertible

Answer: A is singular (det = 0); the system Ax = b has no unique solution.

EXAMPLE 5

Solve 2x + y = 5, 3x + 2y = 8 using the inverse method.

- A = [[2,1],[3,2]]; det = 4−3 = 1. A^{-1} = [[2,−1],[−3,2]]
- x = A^{-1}b = [[2,−1],[−3,2]] · [[5],[8]] = [[10−8],[−15+16]] = [[2],[1]]

Answer: x = 2, y = 1

Guided Practice

Work through each problem below. Use the hint if you get stuck — write your final answer on the last line.

1 Find A^{-1} for A = [[4, 3], [3, 2]].

Hint: Use the 2×2 formula: A^{-1} = (1/det)·[[d,−b],[−c,a]]. Compute det first.

2 Solve Ax = b where A = [[4, 3], [3, 2]] and b = [[1], [2]].

Hint: Use your answer from problem 1: x = A^{-1}b.

3 Find A^{-1} for A = [[1, 0, 2], [0, 1, 3], [0, 0, 1]] using Gauss-Jordan.

Hint: Form [A|I] and row reduce. The upper-triangular structure makes this straightforward.

4 Determine if A = [[3, 6], [1, 2]] is invertible. If not, explain what this means for Ax = b.

Hint: Compute det(A) = 3·2 − 6·1.

5 A company uses the same coefficient matrix A for three different right-hand sides b_1, b_2, b_3. Why is the inverse method more efficient than Gaussian elimination in this case?

Hint: Think about how many times you need to compute A^{-1} vs. how many row reductions you need.

Key Vocabulary

Invertible Matrix	A square matrix A with det(A) ≠ 0; has a unique inverse A^{-1}.
Singular Matrix	A square matrix with det(A) = 0; has no inverse.
Determinant	For A=[[a,b],[c,d]], det(A)=ad−bc; measures whether A is invertible.
Inverse A^{-1}	The matrix satisfying $A\cdot A^{-1} = A^{-1}\cdot A = I$.
Gauss-Jordan (Inv.)	Augment A with I, row reduce to get $[I \mid A^{-1}]$.
Matrix Equation	Ax = b, solved as $x = A^{-1}b$ when A is invertible.

Quick Check Quiz

Circle the letter of the correct answer for each question.

1 **For A = [[2, 1], [5, 3]], det(A) =** Multiple Choice

A 1
B −1
C 6
D 11

2 **If det(A) = 0, then:** Multiple Choice

A $A^{-1} = 0$
B $A^{-1} = I$
C A has no inverse
D A = I

3 **To solve Ax = b using the inverse, you compute:** Multiple Choice

A $x = bA^{-1}$
B $x = A^{-1}b$
C $x = Ab^{-1}$
D x = b/A

4 **A^{-1} for A = [[3, 1], [5, 2]] is:** Multiple Choice

A [[2,−1],[−5,3]]
B [[2,1],[5,3]]
C [[−2,1],[5,−3]]
D [[3,−1],[−5,2]]

5 **The Gauss-Jordan method for finding A^{-1} starts with:** Multiple Choice

- **A** [A|0]
- **B** [I|A]
- **C** [A|I]
- **D** [A|A]

Common Mistakes to Avoid

✗ Computing $x = bA^{-1}$ instead of $x = A^{-1}b$.

✓ Matrix multiplication is not commutative. The correct formula is $x = A^{-1}b$ (A^{-1} on the LEFT). bA^{-1} may not even be defined.

✗ Forgetting to check det(A) before computing the inverse.

✓ Always check det(A) first. If det(A) = 0, the matrix is singular and has no inverse — the method fails.

✗ Applying the 2×2 inverse formula to a 3×3 matrix.

✓ The formula $A^{-1} = (1/\det)\cdot[[d,-b],[-c,a]]$ only works for 2×2 matrices. Use Gauss-Jordan for larger matrices.

✗ Not verifying $A\cdot A^{-1} = I$ after computing the inverse.

✓ Always verify by computing $A\cdot A^{-1}$. If you don't get the identity matrix, there's an arithmetic error.

Math Tips

★ The 2×2 inverse formula: swap the diagonal entries, negate the off-diagonal entries, divide by the determinant.

★ The inverse method shines when you need to solve Ax = b for MULTIPLE right-hand sides b. Compute A^{-1} once, then multiply each b.

★ For a 2×2 system, the inverse method is often faster than Gaussian elimination. For 3×3 and larger, Gaussian elimination is usually more efficient.

★ det(A) = 0 means the rows (or columns) of A are linearly dependent — the system either has no solution or infinitely many.

★ Gauss-Jordan for the inverse: $[A|I] \to [I|A^{-1}]$. The row operations that turn A into I automatically turn I into A^{-1}.